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THE ENERGY RELEASE RATE IN QUASI-STATIC CRACK PROPAGATION AND J-INTEGRAL

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Abstract—The energy release rate and its relation with J-integral is considered from some general point of view.

1. INTRODUCTION

This work is motivated by, and mainly based on, Gurtin's paper[1]. For the sake of clarity and comparison of the results we are considering here the problem of quasi-static (edge) crack propagation in two-dimensional bodies as Gurtin did[1]. We are also using the same notation as in Ref. [1]. The approach is a general one and points out the importance of Euler–Lagrange equations in Gurtin's approach. The application of general results on the special class of materials is given for a micropolar elastic material, elastic dielectric and hyperelastic body.

2. GENERAL STATEMENTS AND DEFINITIONS

We assume that body B contains an edge crack of negligible thickness. The mathematical model of crack is defined as a smooth curve $\mathbf{z}(s)$, where s is an arc length $(0 \leqslant s \leqslant l)$ as shown in Fig. 1; $s = 0$ corresponds to the intersection of the crack with the regular boundary ∂B of the body B; $s = l$ defines the crack tip propagating with time. The crack curve is analytically defined by

$$s(l) = \{\mathbf{z}(s) : 0 \leqslant s \leqslant l\},$$

and the crack tip by

$$\mathbf{z}_l = \mathbf{z}(l).$$

The direction of crack propagation is defined by the unit vector

$$\mathbf{e}(l) = \frac{\partial \mathbf{z}_e}{\partial l}. \tag{2.1}$$

We denote $\mathbf{m}(s)$ the unit vector normal to the crack.

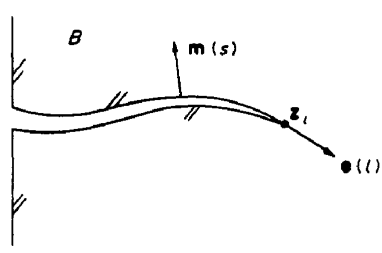

Fig. 1.

We consider the crack which is initially propagating in the sense that $l > 0$ and $\mathrm{d}l/\mathrm{d}t > 0$ at $t = 0$. Consequently $\mathrm{d}l/\mathrm{d}t > 0$ for all t in an interval $[0, t_1)$, l being a strictly increasing function of t for $0 \leqslant t \leqslant t_1$, or equivalently, for l in an interval $[l_0, l_1)$. If we confine our attention to this interval we may use l as our time scale.

The fields $\varphi(\mathbf{x}, l)$, of interest, are defined at $\forall \mathbf{x} \in B - s(l)$ $\forall l \in [l_0, l_1)$. Such fields are C^n fracture fields ($n > 0$ an integer) if

(i) the derivatives of φ of order $\leqslant n$ exist away from the crack;
(ii) φ and its derivatives of order $\leqslant n$ are continuous away from the crack and, except at the tip, are continuous up to the crack from either side.

Thus $\varphi(\mathbf{x}, l)$, as a function of $\mathbf{x}$ suffers (at most) a jump discontinuity across $\varphi(l) - \mathbf{z}_l$.

The values of φ on the upper and lower faces of the crack can be denoted as

$$\varphi^{\pm}(s, l) = \lim_{\substack{\delta \to 0 \\ (0 \leqslant s \leqslant l)}} \varphi[\mathbf{z}(s) \pm \delta \mathbf{m}(s); l]. \tag{2.2}$$

It is important to note that in the above definition nothing is said about the behaviour of function φ at the crack tip. In fact, φ is singular at the crack tip if identified with the stress or stored energy.

Therefore we isolate the crack tip with a disc D_δ, δ being the radius of a circle centred at the crack tip. The disc moves with the crack tip, and

$$B_\delta(l) = B - D_\delta(l).$$

Further, $\mathbf{n}$ will always denote the outward unit normal to ∂D_δ, while $\mathbf{v}$ denotes the outward unit normal to $2B$ (Fig. 2).

The position of a material point can be defined relative to the fixed point 0 or to the crack tip. Thus

$$\mathbf{r} = \mathbf{x} - \mathbf{z}_l, \tag{2.3}$$

enabling the field $\varphi(\mathbf{x}, l)$ to be expressed through $\mathbf{r}$

$$\varphi(\mathbf{x}, l) = \varphi(\mathbf{r} + \mathbf{z}_l, l) \equiv \psi(\mathbf{r}, l). \tag{2.4}$$

We write

$$\dot{\varphi} = \frac{\partial \varphi}{\partial l}, \tag{2.5}$$

$$\mathrm{f}(\mathbf{x}, l) \equiv \left.\frac{\partial \psi}{\partial l}\right|_{\mathbf{r} = \mathbf{x} - \mathbf{z}_l}, \tag{2.6}$$

with the partial derivative of φ and ψ with respect to l holding $\mathbf{x}$ and $\mathbf{r}$ fixed, respectively, and $\nabla\varphi$ for the gradient of φ with respect to $\mathbf{x}$, i.e. $\nabla\psi$ for the gradient of ψ with respect to $\mathbf{r}$.

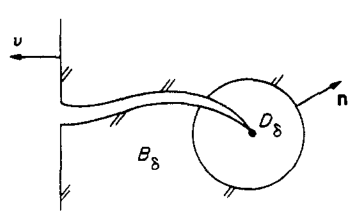

Fig. 2.

3. EULER–LAGRANGE EQUATION ASSUMPTIONS

So defined crack fields may be tensor fields of any order and may have different physical meaning. They may represent the arguments of some function, say, H.

Let the function

$$H = H(\underset{a}{\mathbf{w}}, \nabla\underset{a}{\mathbf{w}}, \underset{b}{\varphi}, \nabla\underset{b}{\varphi}), \tag{3.1}$$

defined on B_δ, be a continuous and differentiable function of its arguments

$$\begin{aligned} \underset{a}{\mathbf{w}} &= \underset{a}{\mathbf{w}}(\mathbf{x}, l), \quad (a = 1, 2, \ldots, n), \\ \underset{b}{\varphi} &= \underset{b}{\varphi}(\mathbf{x}, l), \quad (b = 1, 2, \ldots, m), \end{aligned} \tag{3.2}$$

which are also continuous and differentiable functions of the variables $(\mathbf{x}, l)$. Thus

$$\begin{aligned} \dot{H} &= \frac{\partial H}{\partial \underset{a}{\mathbf{w}}}\dot{\underset{a}{\mathbf{w}}} + \frac{\partial H}{\partial \nabla\underset{a}{\mathbf{w}}}\nabla\dot{\underset{a}{\mathbf{w}}} + \frac{\partial H}{\partial \underset{b}{\varphi}} + \frac{\partial H}{\partial \nabla\underset{b}{\varphi}}\nabla\dot{\underset{b}{\varphi}} \\ &= \left[\frac{\partial H}{\partial \underset{a}{\mathbf{w}}} - \nabla\left(\frac{\partial H}{\partial \nabla\underset{a}{\mathbf{w}}}\right)\right]\dot{\underset{a}{\mathbf{w}}} + \left[\frac{\partial H}{\partial \underset{b}{\varphi}} - \nabla\left(\frac{\partial H}{\partial \nabla\underset{b}{\varphi}}\right)\right]\dot{\underset{b}{\varphi}} + \nabla\left(\frac{\partial H}{\partial \nabla\underset{a}{\mathbf{w}}}\dot{\underset{a}{\mathbf{w}}} + \frac{\partial H}{\partial \nabla\underset{b}{\varphi}}\dot{\underset{b}{\varphi}}\right), \end{aligned}$$

where (and throughout this paper) we employ the usual summation convention for repeated suffixes.

The expressions in square brackets represent Euler–Lagrange equations for the function H with respect to its arguments when they are equal to zero, i.e.

$$\begin{aligned} \nabla\left(\frac{\partial \mathrm{H}}{\partial \nabla\underset{a}{\mathbf{w}}}\right) - \frac{\partial H}{\partial \underset{a}{\mathbf{w}}} &= 0, \\ \nabla\left(\frac{\partial H}{\partial \nabla\underset{b}{\varphi}}\right) - \frac{\partial H}{\partial \underset{b}{\varphi}} &= 0. \end{aligned} \tag{3.3}$$

In this case the previous expression reduces to the much simpler form

$$\dot{H} = \nabla(\underset{a}{\mathbf{S}} \cdot \dot{\underset{a}{\mathbf{w}}} + \underset{b}{\phi}\dot{\underset{b}{\varphi}}), \tag{3.4}$$

which are by definition

$$\begin{aligned} \underset{a}{\mathbf{S}} &= \frac{\partial H}{\partial \nabla\underset{a}{\mathbf{w}}}, \\ \underset{b}{\phi} &= \frac{\partial H}{\partial \nabla\underset{b}{\varphi}}. \end{aligned} \tag{3.5}$$

Now we state Gurtin's hypotheses[1] concerning the fields $\underset{a}{\mathbf{w}}(\mathbf{x}, l)$, $\underset{b}{\varphi}(\mathbf{x}, l)$ and $H(\mathbf{x}, l)$ (the last being the symbol for $H(\underset{a}{\mathbf{w}}, \nabla\underset{a}{\mathbf{w}}, \underset{b}{\varphi}, \nabla\underset{b}{\varphi})$):

(A_1) $\underset{a}{\mathbf{w}}$ and $\underset{b}{\varphi}$ are C^2 fracture fields.

(A_2) $\underset{a}{\mathbf{w}}$, $\underset{b}{\varphi}$, $\underset{a}{\mathbf{S}}$, $\underset{b}{\phi}$ and H satisfy equations (3.3) and (3.5) away from the crack (consequently the fields $\underset{a}{\mathbf{S}}$, $\underset{b}{\phi}$ and H are necessarily C^1 fracture fields).

(A_3) For each l, $H(\mathbf{x}, l)$ is integrable in $\mathbf{x}$ over B.

(A_4) As $\delta \to 0$

$$\mathbf{e}\cdot\int_{\partial D_\delta} H\mathbf{n}\;\mathrm{d}s,\quad \int_{\partial D_\delta} \underset{a}{\mathbf{S}}\mathbf{n}\cdot\underset{a}{\dot{\mathbf{w}}}\;\mathrm{d}s,\quad \int_{\partial D_\delta} \underset{b}{\phi}\mathbf{n}\cdot\underset{b}{\varphi}\;\mathrm{d}s,$$

converge uniformly in l to their limits.

(A_5) For crack surfaces

$$\left.\begin{aligned}\underset{a}{\mathbf{S}}^{\pm}(s\,;l)\mathbf{m}(s) &= 0\\ \underset{b}{\phi}^{\pm}(s\,;l)\mathbf{m}(s) &= 0\end{aligned}\right\}\quad (0\leqslant s\leqslant l).$$

(A_6) As

$$\left.\begin{aligned}&\int_{\partial D_\delta}\underset{a}{\mathbf{S}}\mathbf{n}\;\mathrm{d}s\to 0,\quad \int_{\partial D_\delta}|\underset{a}{\mathbf{S}}\mathbf{n}|\;\mathrm{d}s\\ &\int_{\partial D_\delta}\underset{b}{\phi}\mathbf{n}\;\mathrm{d}s\to 0,\quad \int_{\partial D_\delta}|\underset{a}{\phi}\mathbf{n}|\;\mathrm{d}s\end{aligned}\right\}\ \text{remain bounded.}$$

To discuss further hypotheses we consider eqn (2.6) in detail. In the case of the identification of the fields φ with, e.g. displacement field or microgyration vector field, guided by experience, we expect singular behaviour of the gradient $\nabla\psi(\mathbf{r}, l)$ with respect to $\mathbf{r}$, but continuity in the partial derivatives (2.7) with respect to l; more precisely, we assume that

(A_7) f is continuous at the crack tip, i.e. there exists a function $\mathrm{f}_T(l)$ such that $\mathrm{f}(\mathbf{x}, l)\to \mathrm{f}_T(l)$, whenever $\mathbf{x}$ tends towards the crack tip from $B-s(l)$ or equivalently, such that

$$\sup_{\mathbf{x}\in D_\delta(l)}|\mathrm{f}(\mathbf{x}, l)\mathrm{f}_T(l)|\to 0,\qquad \text{when } \delta\to 0.$$

In our case we identify the field φ with $\underset{a}{\mathbf{w}}(\mathbf{x}, l)$ and $\underset{b}{\varphi}(\mathbf{x}, l)$; $\mathrm{f}(\mathbf{x}, l)$ with the derivatives of these fields in the sense of paragraph 2.

Physically they may be displacement field, microgyration field, electric field, electric potential and so on. Then the function H may represent stored energy function, Helmholtz free energy or electric enthalpy and (3.5) some constitutive relations. In these cases an analysis of the aforementioned hypotheses enables us to conclude: (A_4), (A_6) and (A_7) limit the strength of the singularity at the crack tip. In particular, (A_6) precludes the possibility of a concentrated force or moment at the crack tip. Using these hypotheses we are now in a position to prove the following:

proposition:

$$\frac{\mathrm{d}}{\mathrm{d}l}\int_{B_\delta} H\;\mathrm{d}a+\mathbf{e}\cdot\int_{\partial D_\delta} H\mathbf{n}\;\mathrm{d}s = \int_{\partial B}(\underset{a}{\mathbf{S}}\mathbf{v}\cdot\underset{a}{\dot{\mathbf{w}}}+\underset{b}{\phi}\mathbf{v}\underset{b}{\dot{\varphi}})\;\mathrm{d}s-\int_{\partial D_\delta}(\underset{a}{\mathbf{S}}\mathbf{n}\cdot\mathbf{w}+\underset{b}{\phi}\mathbf{n}\underset{b}{\dot{\varphi}})\;\mathrm{d}s. \tag{3.6}$$

Proof. As a consequence of (A_2) expression (3.4) holds. Applying the divergence theorem to the region $B_\delta(l)$ whose boundary consists of ∂B, ∂D_δ and the two faces of the appropriate portion of the crack on this expression, in view of (A_5), we obtain:

$$\int_{B_\delta}\dot{H}\;\mathrm{d}a = \int_{\partial B}(\underset{a}{\mathbf{S}}\mathbf{v}\cdot\underset{a}{\dot{\mathbf{w}}}+\underset{b}{\phi}\mathbf{v}\underset{b}{\dot{\varphi}})\;\mathrm{d}s-\int_{\partial D_\delta}(\underset{a}{\mathbf{S}}\mathbf{n}\cdot\underset{a}{\dot{\mathbf{w}}}+\underset{b}{\phi}\mathbf{n}\underset{b}{\dot{\varphi}})\;\mathrm{d}s.$$

On the other hand, according a classical transport theorem we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_\delta} H\,\mathrm{d}a = \int_{B_\delta} \dot{H}\,\mathrm{d}a - \mathbf{e}\cdot\int_{\partial D_\delta} H\mathbf{n}\,\mathrm{d}s.$$

The last two relations imply (3.6).

Corollary: The derivatives exists:

$$\frac{\mathrm{d}}{\mathrm{d}l}\int_{B} H\,\mathrm{d}a, \tag{3.7}$$

while

$$\lim_{\delta\to 0}\frac{\mathrm{d}}{\mathrm{d}l}\int_{B_\delta} H\,\mathrm{d}a = \frac{\mathrm{d}}{\mathrm{d}l}\int_{B} H\,\mathrm{d}a. \tag{3.8}$$

Proof: By (3.6) and (A_4)

$$\frac{\mathrm{d}}{\mathrm{d}l}\int_{B_\delta} H\,\mathrm{d}a$$

converges uniformly in l to a limit, also, by (A_3)

$$\int_{B_\delta} H\,\mathrm{d}a \to \int_{B} H\,\mathrm{d}a.$$

The last two observations yield the conclusions (3.7) and (3.8).

4. THE ENERGY RELEASE RATE

Let H represent the energy function, and $\underset{a}{\mathbf{w}}$ and $\underset{b}{\varphi}$ fields characterizing the motion of a deformed body. Then we call the function ε, defined as

$$\varepsilon(l) = -\frac{\mathrm{d}}{\mathrm{d}l}\int_{B} H\,\mathrm{d}a + \int_{\partial B}(\underset{a}{\mathbf{S}}\mathbf{v}\cdot\underset{a}{\mathbf{w}} + \underset{b}{\phi}\mathbf{v}\underset{b}{\dot{\varphi}})\,\mathrm{d}s, \tag{4.1}$$

the energy release rate. By rewriting (4.1) in the form

$$\frac{\mathrm{d}}{\mathrm{d}l}\int_{B} H\,\mathrm{d}a + \varepsilon(l) = \int_{\partial B}(\underset{a}{\mathbf{S}}\mathbf{v}\cdot\underset{a}{\dot{\mathbf{w}}} + \underset{b}{\phi}\mathbf{v}\underset{b}{\dot{\varphi}})\,\mathrm{d}s,$$

it is possible to interpret $\varepsilon(l)$ as the (net) rate of increase of crack energy.

As the direct consequences of (3.6), (3.8) and (4.1) holds the following

Theorem:

$$\varepsilon = \lim_{\delta\to 0}\int_{\partial D_\delta}(H\mathbf{n}\cdot\mathbf{e} + \underset{a}{\mathbf{S}}\mathbf{n}\cdot\underset{a}{\dot{\mathbf{w}}} + \underset{b}{\phi}\mathbf{n}\underset{b}{\varphi})\,\mathrm{d}s. \tag{4.2}$$

The term

$$\int_{\partial D_\delta} H\mathbf{n}\cdot\mathbf{e}\,\mathrm{d}s,$$

represents the energy flow out of B_δ as a consequence of the motion of D_δ with the tip. Therefore, the right-hand side of (4.2) gives the energy released by the body and absorbed by the moving crack tip plus the power expended by the body at the tip.

We may write (4.2) in an alternative form according to the theorem:

$$\lim_{\delta\to 0}\int_{\partial D_\delta} \underset{a}{\mathbf{S}}\mathbf{n}\cdot\underset{a}{\dot{\mathbf{w}}}\,\mathrm{d}s = -\lim_{\delta\to 0}\mathbf{e}\cdot\int_{\partial D_\delta}\nabla\underset{a}{\mathbf{w}}^T\underset{a}{\mathbf{S}}\mathbf{n}\,\mathrm{d}s, \tag{4.3}$$

$$\lim_{\delta\to 0}\int_{\partial D_\delta}\underset{b}{\phi}\mathbf{n}\underset{b}{\dot{\varphi}}\,\mathrm{d}s = -\lim_{\delta\to 0}\int_{\partial D_\delta}\nabla\underset{b}{\varphi}\underset{b}{\phi}\mathbf{n}\,\mathrm{d}s. \tag{4.4}$$

Proof: According to (2.3), (2.4) and (2.1) we have

$$\varphi(\mathbf{x},l) = \psi(\mathbf{x}-\mathbf{z}_l,l),$$

$$\left.\frac{\partial\mathbf{r}}{\partial l}\right|_{\mathbf{x}} = -\mathbf{e}(l).$$

so that

$$\dot{\varphi}(\mathbf{x},l) = -\nabla\psi(\mathbf{x}-\mathbf{z}_l,l)\cdot\mathbf{e}(l)+f(\mathbf{x},l).$$

But

$$\nabla\varphi(\mathbf{x},l)=\nabla\psi(\mathbf{x}-\mathbf{z}_l,l),$$

hence

$$\dot{\varphi} = -(\nabla\varphi)\cdot\mathbf{e}+\mathrm{f}. \tag{4.5}$$

If we identify φ with $\underset{a}{\mathbf{w}}$, or with $\underset{b}{\dot{\varphi}}$, we have

$$\underset{a}{\dot{\mathbf{w}}} = -(\nabla\underset{a}{\mathbf{w}})\cdot\mathbf{e}+\underset{a}{\mathbf{f}}, \tag{4.6}$$

$$\underset{b}{\dot{\varphi}} = -(\nabla\underset{b}{\varphi})\cdot\mathbf{e}+\underset{b}{\mathrm{f}}, \tag{4.7}$$

where $\underset{a}{\mathbf{f}}$, or $\underset{b}{\mathrm{f}}$, are defined according to (2.6). Next, by (A_6) and (A_7),

$$\left|\int_{\partial D_\delta}\underset{a}{\mathbf{S}}\cdot\mathbf{n}(\underset{a}{\mathbf{f}}-\underset{a}{\mathbf{f}}_T)\,\mathrm{d}s\right| \leqslant \sup_{D_\delta}\left|\underset{a}{\mathbf{f}}-\underset{a}{\mathbf{f}}_T\right|\int_{\partial D_\delta}\left|\underset{a}{\mathbf{S}}\cdot\mathbf{n}\right|\mathrm{d}s\to 0,$$

$$\int_{\partial D_\delta}\underset{a}{\mathbf{S}}\mathbf{n}\cdot\underset{a}{\mathbf{f}}_T\,\mathrm{d}s = \underset{a}{\mathbf{f}}_T\cdot\int_{\partial D_\delta}\underset{a}{\mathbf{S}}\mathbf{n}\,\mathrm{d}s\to 0,$$

$$\int_{\partial D_\delta}\underset{a}{\mathbf{S}}\mathbf{n}\cdot\underset{a}{\mathbf{f}}\,\mathrm{d}s\to 0.$$

These results together with (4.6) yield (4.3). The proof of (4.4) is strictly analogous. The following corollary applies to the last two theorems.

Corollary:

$$\varepsilon = \lim_{\delta\to 0} \mathbf{e}\cdot \int_{\partial D_\delta} (H\mathbf{n} - \nabla \underset{a}{\mathbf{w}}^T\cdot \underset{a}{\mathbf{S}}\mathbf{n} - \nabla\underset{b}{\varphi}\underset{b}{\phi}\mathbf{n})\,\mathrm{d}s. \tag{4.8}$$

From this expression we see that $\varepsilon(l)$ can be computed from a knowledge of fields: $\underset{a}{\mathbf{w}}(\mathbf{x}, l)$ and $\underset{b}{\varphi}(\mathbf{x}, l)$ at the given crack length l in contrast to (4.1) whose computation involves $\underset{a}{\mathbf{w}}(\mathbf{x}, l)$ and $\underset{b}{\varphi}(\mathbf{x}, l)$ for crack lengths τ close to t.

5. J-INTEGRAL

Let G denote any smooth non-selfintersecting path, which begins and ends on the crack and which surrounds the crack tip. Let $\mathbf{n}$ denote the unit vector normal on G directed away from the tip. Then we call the expression

$$J(G) \equiv \mathbf{e}\cdot \int_G (H\mathbf{n} - \nabla \underset{a}{\mathbf{w}}^T\cdot \underset{a}{\mathbf{S}}\mathbf{n} - \nabla\underset{b}{\varphi}\underset{b}{\phi}\mathbf{n})\,\mathrm{d}s \tag{5.1}$$

the J-integral for the path G (Fig. 3).

Theorem:

$$\varepsilon = \lim_{\delta\to 0} J(\partial D_\delta). \tag{5.2}$$

Further, if the crack is straight, then $J(G)$ is independent of the path G surrounding the crack tip and

$$\varepsilon = J(G). \tag{5.2a}$$

Proof: Equation (5.2) is a direct consequence of (4.8) and (5.1). The remainder of the proof is based on the extended Eshelby's relation[2]

$$\operatorname{div}(H\mathbf{I} - \nabla\underset{a}{\mathbf{w}}^T\underset{a}{\mathbf{S}} - \nabla\underset{b}{\varphi}\underset{b}{\phi}) = 0, \tag{5.3}$$

where $\mathbf{I}$ is the unit tensor. This relation is proved in an appendix of this paper.

Assume now that the crack is straight. Choose G; let δ be small enough that G contains ∂D_δ, and denote by F_δ the region between G and ∂D_δ. Then F_δ is bounded with G, ∂D_δ and the two faces of the crack. On the faces of the crack the normal $\mathbf{m}$ is perpendicular to $\mathbf{l}$.

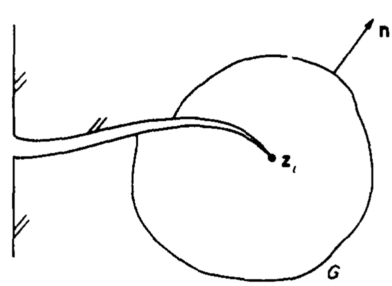

Fig. 3.

If we integrate now (5.3) over F_δ and make use of the divergence theorem and (5.1), we arrive at the relation:

$$J(G) = J(\partial D_\delta),$$

which yields, according to (5.2), (5.2a). Q.E.D.

Now we are proceeding to the application of so derived general results on the special class of material.

6. MICROPOLAR ELASTIC CONTINUUM†

The general theory of simple micro-elastic solids has been formulated by Eringen and Suhubi[3]. When body forces and body couples are absent the equations of equilibrium reduce to

$$t^{kl}{}_{,k} = 0, \tag{6.1}$$

$$m^{kl}{}_{,k} + \varepsilon^{lmn} t_{mn} = 0, \tag{6.2}$$

where t^{kl} is the stress tensor, m^{kl} is the couple stress tensor and ε^{lmn} is Ricci's tensor.

The constitutive equations for the elastic micropolar continuum are given by[4]

$$t^l_k = \rho x^l_{;K} \frac{\partial \psi}{\partial x^k_{;K}}, \tag{6.3}$$

$$m^l_k = \rho x^l_{;K} \Lambda^{-1n}_{.k} \frac{\partial \psi}{\partial \varphi^n_{;K}}, \tag{6.4}$$

where $\psi = \psi(x^k_{;K}, \varphi, \varphi^k_{;K})$ is the free energy density, ρ is the mass density, $x^k_{;K} (\equiv (\partial x^k / \partial X^K))$ is the deformation gradient and φ^n is the microrotation vector. The quantity Λ_{kl} relates the angular velocity vector v_k (the microgyration vector) to the microrotation vector by

$$v_k = \Lambda_{kl} \dot{\varphi}^l.$$

For the further manipulation we use the material equation of continuity

$$\rho_0 = \rho \mathscr{J} \tag{6.5}$$

and the equation

$$\rho \frac{\partial \psi}{\partial \varphi^n} + \varepsilon_{rkl} t^{kl} \Lambda^r_{.k} - m^l_{.k} \Lambda^k_{.n,l} = 0, \tag{6.6}$$

which contains the polar statement of objectivity, as well as the relations

$$\begin{gathered} \mathscr{J} = \sqrt{\frac{g}{G}} j; \quad j = |x^k_{;K}|; \\ (\mathscr{J} X^K_{;k})_{;K} = 0; \quad (\mathscr{J}^{-1} x^k_{;K})_{;k} = 0; \\ \Lambda^k_{.n} \Lambda^{-1n}_{.l} = \delta^k_l, \end{gathered} \tag{6.7}$$

† In this paragraph, for the sake of a generality, we use the spatial x^k and the material coordinates X^K, respectively, whereby all quantities are expressed as functions of X^K and t.

where ρ_0 is the mass density in the reference configuration and g and G are the determinants of the metric tensors in the reference and the present configurations, respectively.

By "," and ";" we denote a partial and total covariant derivative respectively. The relations and the expressions (6.1–7) are given in the paper[4].

We demonstrate now, by making use of the deformation energy Σ, defined by

$$\Sigma = \rho_0\psi \tag{6.8}$$

that the equilibrium conditions (6.3) and (6.4) can be written in the form of the Euler–Lagrange's eqns (3.3). First we write

$$t^l_{.k,l} = \left(\rho x^l_{;K}\frac{\partial\psi}{\partial x^k_{;K}}\right)_{;l} = \left(\mathscr{J}^{-1}x^l_{;K}\frac{\partial\Sigma}{\partial x^k_{;K}}\right)_{;l} = \mathscr{J}^{-1}x^l_{;K}\left(\frac{\partial\Sigma}{\partial x^k_{;K}}\right)_{;l} = \mathscr{J}^{-1}\left(\frac{\partial\Sigma}{\partial x^k_{;K}}\right)_{;K},$$

using (6.3), (6.5), (6.8) and $(6.7)_4$. Then (6.1) becomes

$$\left(\frac{\partial\Sigma}{\partial x^k_{;K}}\right)_{;K} = 0. \tag{6.9}$$

These Euler–Lagrange's equations are well known in the theory of elasticity of non-polar continuum as the special case of (3.3), when H, for some fixed a does not depend on $\overset{a}{\mathbf{w}}$.

The equilibrium conditions (6.4) we treat in the following way: we multiply them by $\Lambda^r_{.n}$ so that

$$m^l_{.r,l}\Lambda^r_{.n} + \varepsilon_{rkl}t^{kl}\Lambda^r_{.n} = 0.$$

These equations and (6.6) yield to

$$\rho\frac{\partial\psi}{\partial\varphi^n} - m^l_{.r,l}\Lambda^r_{.n} - m^l_{.r}\Lambda^r_{.n,l} = 0$$

or

$$\rho\frac{\partial\psi}{\partial\varphi^n} - (m^l_{.r}\Lambda^r_{.n})_{,l} = 0. \tag{6.10}$$

Further, from (6.4) and $(6.7)_5$ we have

$$m^l_{.r}\Lambda^r_{.n} = \rho x^k_{;K}\frac{\partial\psi}{\partial\varphi^k_{;K}}.$$

Then

$$(m^l_{.r}\Lambda^r_{.n})_{,l} = \left(\rho x^k_{;K}\frac{\partial\psi}{\partial\varphi^n_{;K}}\right)_{;l} = \mathscr{J}^{-1}x^l_{;K}\left(\frac{\partial\Sigma}{\partial\varphi^n_{;K}}\right)_{;l} = \mathscr{J}^{-1}\left(\frac{\partial\Sigma}{\partial\varphi^n_{;K}}\right)_{;K},$$

where we have used (6.5), (6.8) and $(6.7)_4$. Substituting this expression in (6.10), multiplying the subsequent relation by J and using (6.5) and (6.8) we arrive at the Euler–Lagrange's equations

$$\frac{\partial\Sigma}{\partial\varphi^n} - \left(\frac{\partial\Sigma}{\partial\varphi^n_{;K}}\right)_{;K} = 0. \tag{6.11}$$

It is not difficult to show that eqns (6.9) and (6.11) hold also in the case when micropolar continuum is two-dimensional. In that case some of the expressions, we used, may be simplified; it is obvious that then all indices run from 1 to 2. We do not do it indicating that so derived result, under aforementioned assumptions, hold generally. Hence, the function $\Sigma = \Sigma(x^k_{;K}; \varphi^n; \varphi^n_{;K})$ satisfies Euler–Lagrange's equations (6.9) and (6.11), enabling an application of our results, namely J-integral, in the case of a cracked micropolar elastic continuum.

To this aim we identify $\nabla \underset{1}{\mathbf{w}}$ with $\mathbf{F}\{x^k_{;K}\}$, $\underset{2}{\mathbf{w}}$ with $\boldsymbol{\varphi}\{\varphi^n\}$ and H with Σ. Then, by (3.5), the quantity $\underset{1}{\mathbf{S}}$ is identified by

$$\frac{\partial \Sigma}{\partial \mathbf{F}}\left(\frac{\partial \Sigma}{\partial x^k_{;K}}\right),$$

as well as $\underset{2}{\mathbf{S}}$ by

$$\frac{\partial \Sigma}{\partial \nabla \boldsymbol{\varphi}}\left(\frac{\partial \Sigma}{\partial \varphi^n_{;K}}\right).$$

On the other hand, using (6.8), (6.3), (6.5) and $(6.7)_5$ one arrives at

$$\frac{\partial \Sigma}{\partial x^k_{;K}} = \mathscr{J} X^K_{;l} t^l_{.k}, \qquad \frac{\partial \Sigma}{\partial \varphi^n_{;K}} = \mathscr{J} X^K_{;l} \Lambda^k_{.n} m^l_{.k}.$$

The right hand sides of these relations represent stress and couple stress tensors in the reference configuration. More precisely, the first one represents the first Piola–Kirchhoff tensor $T^K_{.k}$, and the latter is a counterpart for couple stress $M^K_{.n}$, i.e.

$$T^K_{.k} = \frac{\partial \Sigma}{\partial x^k_{;K}}; \qquad M^K_{.k} = \frac{\partial \Sigma}{\partial \varphi^k_{;K}}, \tag{6.12}$$

or

$$\begin{aligned} \mathbf{t} = \{T^K_{.k}\} &= \frac{\partial \Sigma}{\partial \mathbf{F}}, \\ \mathbf{M} = \{M^K_{.k}\} &= \frac{\partial \Sigma}{\partial \nabla \boldsymbol{\varphi}}. \end{aligned} \tag{6.13}$$

The J-integral may now be, for this case, written in the form

$$J(G) = \mathbf{e} \cdot \int_G (\Sigma \mathbf{n} - \mathbf{F}^T \cdot \mathbf{T}\mathbf{n} - \nabla \boldsymbol{\varphi} \cdot \mathbf{M}\mathbf{n}) \, \mathrm{d}s. \tag{6.14}$$

The relations (5.2) and (5.2a) are also valid under the aforementioned conditions.

J-Integral (6.14) may be written through the displacement gradient if we take into account that

$$\mathbf{x} = \mathbf{U} + \mathbf{X},$$

and

$$\mathbf{F} = \nabla \mathbf{U} + \mathbf{I}. \tag{6.15}$$

Using this relation it is possible to express Σ as a function of $\nabla\mathbf{U}$. Then (6.13) becomes

$$\mathbf{T} = \frac{\partial \Sigma}{\partial \nabla \mathbf{U}}. \tag{6.16}$$

Identifying $\underset{a}{\nabla\mathbf{w}}$ with $\nabla\mathbf{U}$, (6.14) becomes

$$J(G) = \mathbf{e}\cdot\int_G (\Sigma\mathbf{n} - \nabla\mathbf{U}^T\mathbf{T}\mathbf{n} - \nabla\varphi^T\mathbf{M}\mathbf{n})\,\mathrm{d}s, \tag{6.17}$$

according to (5.1). For the special case of the non-polar continuum Σ is independent on φ and $\nabla\varphi$, find the couple stress $\mathbf{M}$ is zero, as it is directly seen from (6.12). Then

$$J(G) = \mathbf{e}\cdot\int_G (\Sigma\mathbf{n} - \nabla\mathbf{U}^T\mathbf{T}\mathbf{n})\,\mathrm{d}s,$$

reduces to Gurtin's result[1] for the two dimensional body.

7. CLASSICAL, LINEAR THEORY OF ELASTIC DIELECTRICS

In Mindlin's paper[5] devoted to elasticity, piezoelectricity and crystal lattice dynamics, extended Toupin's variational principle is used to account for the contribution of the polarization gradient. Using an electric enthalpy H, defined by

$$H = W^L(e_{ij};\pi_i;\pi_{j,i}) - \tfrac{1}{2}\varepsilon_0\varphi_{,i}\varphi_{,i} + \varphi_{,i}\pi_i, \tag{7.1}$$

where the above quantities represent W^L stored energy of deformation and polarization, $e_{ij} = U_{(i,j)}$ infinitesimal strain tensor, π_i polarization vector, ε_0 permitivity of vacuum and φ electric potential, the following Euler—Lagrange equations for the case of equilibrium and in the absence of body force and external electric field are derived

$$\begin{aligned} t_{ij,i} &= 0,\\ E_{ij,i} + E_j^L - \varphi_{,j} &= 0,\\ -\varepsilon_0\varphi_{,jj} + \pi_{j,j} &= 0. \end{aligned} \tag{7.2}$$

The stress tensor t_{ij}, the local electric stress tensor E_{ij} and the electric field E_j^L are linked with W^L by the following relations:

$$t_{ij} = \frac{\partial W^L}{\partial e_{ij}}, \quad E_{ij} = \frac{\partial W^L}{\partial \pi_{j,i}}, \quad E_i^L = -\frac{\partial W^L}{\partial \pi_i}. \tag{7.3}$$

Indeed, it is easy to show directly that (7.2) represent Euler–Lagrange equations for H since the following relations hold

$$\begin{aligned} &\frac{\partial H}{\partial U_{i,j}} = \frac{\partial H}{\partial e_{ij}} = \frac{\partial W^L}{\partial e_{ij}},\\ &\frac{\partial H}{\partial \pi_{j,i}} = \frac{\partial W^L}{\partial \pi_{j,i}},\\ &\frac{\partial H}{\partial \pi_i} = \frac{\partial W^L}{\partial \pi_i} + \varphi_{,i},\\ &\frac{\partial H}{\partial \varphi_{,i}} = -\varepsilon_0\varphi_{,i} + \pi_i\\ &\frac{\partial H}{\partial x_i} = 0;\quad \frac{\partial H}{\partial U_i} = 0;\quad \frac{\partial H}{\partial \varphi} = 0, \end{aligned} \tag{7.4}$$

as can be seen from (7.1). Then, making use of (7.3) and (7.4), eqns (7.2) can be written as

$$
\begin{aligned}
&\left(\frac{\partial H}{\partial U_{j,i}}\right)_{,i} = 0,\\
&\left(\frac{\partial H}{\partial \pi_{j,i}}\right)_{,i} - \frac{\partial H}{\partial \pi_j} = 0, \qquad (7.5)\\
&\left(\frac{\partial H}{\partial \varphi_{,i}}\right)_{,i} = 0.
\end{aligned}
$$

If these equations hold also in two-dimensional case then the general results obtained so far can be applied.

Comparing (3.1) and (7.2) it is obvious that $\nabla \underset{1}{\mathbf{w}}$ may be identified with $\nabla\mathbf{U}(U_{i,j})$, $\underset{2}{\mathbf{w}}$ with $\boldsymbol{\pi}(\pi_i)$, φ with φ. Thus $\underset{1}{\phi}_1 = -\varepsilon_0\nabla\varphi+\boldsymbol{\pi}$, by $(3.5)_2$. Introducing the new symbols

$$\mathbf{t}(t_{ij}); \quad \mathbf{E}^L(E_i^L); \quad \mathbf{E}(E_{ij})$$

(5.1) may be expressed in the form

$$J(G) = \mathbf{e}\cdot\int_G [H\mathbf{n} - \nabla\mathbf{U}^T\mathbf{t}\mathbf{n} - \nabla\boldsymbol{\pi}^T\mathbf{E}\mathbf{n} - \nabla\varphi(-\varepsilon_0\nabla\varphi+\boldsymbol{\pi})\mathbf{n}]\,\mathrm{d}s \qquad (7.6)$$

representing J-integral for the described model of a cracked elastic dielectric. For such body (5.2) and (5.2a) are valid under the aforementioned conditions.

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APPENDIX

Starting from the expression

$$
\begin{aligned}
\mathrm{div}_k(\mathrm{H}\,\mathrm{I}) = [\mathrm{H}\delta_{ik}]_{,k} &= \frac{\partial \mathrm{H}}{\partial \mathrm{x}_i} = \frac{\partial \mathrm{H}}{\partial \underset{a}{\mathrm{W}}_j}\frac{\partial \underset{a}{\mathrm{W}}_j}{\partial \mathrm{x}_i} + \frac{\partial \mathrm{H}}{\partial \underset{a}{\mathrm{W}}_{j,k}}\frac{\partial^2 \underset{a}{\mathrm{W}}_j}{\partial \mathrm{x}_k\,\partial \mathrm{x}_i}\\
&+ \frac{\partial \mathrm{H}}{\partial \underset{b}{\varphi}}\frac{\partial \underset{b}{\varphi}}{\partial \mathrm{x}_i} + \frac{\partial \mathrm{H}}{\partial \underset{b}{\varphi}_{,k}}\frac{\partial^2 \underset{b}{\varphi}}{\partial \mathrm{x}_k\,\partial \mathrm{x}_i} = \frac{\partial \mathrm{H}}{\partial \underset{a}{\mathrm{W}}_j}\frac{\partial \underset{a}{\mathrm{W}}_j}{\partial \mathrm{x}_i} + \frac{\partial}{\partial \mathrm{x}_k}\left(\frac{\partial \mathrm{H}}{\partial \underset{a}{\mathrm{W}}_{j,k}}\frac{\partial \underset{a}{\mathrm{W}}_j}{\partial \mathrm{x}_i}\right)\\
&- \frac{\partial}{\partial \mathrm{x}_k}\frac{\partial \mathrm{H}}{\partial \underset{a}{\mathrm{W}}_{j,k}}\frac{\partial \underset{a}{\mathrm{W}}_j}{\partial \mathrm{x}_i} + \frac{\partial \mathrm{H}}{\partial \underset{b}{\varphi}}\frac{\partial \underset{b}{\varphi}}{\partial \mathrm{x}_i} + \frac{\partial}{\partial \mathrm{x}_k}\left(\frac{\partial \mathrm{H}}{\partial \underset{b}{\varphi}_{,k}}\frac{\partial \underset{b}{\varphi}}{\partial \mathrm{x}_i}\right) - \frac{\partial}{\partial \mathrm{x}_k}\frac{\partial \mathrm{H}}{\partial \underset{b}{\varphi}_{,k}}\frac{\partial \underset{b}{\varphi}}{\partial \mathrm{x}_i}\\
&= \left.\frac{\partial \mathrm{H}}{\partial \underset{a}{\mathrm{W}}_j} - \frac{\partial}{\partial \mathrm{x}_k}\frac{\partial \mathrm{H}}{\partial \underset{a}{\mathrm{W}}_{j,k}}\right]\frac{\partial \underset{a}{\mathrm{W}}_j}{\partial \mathrm{x}_i} + \left[\frac{\partial \mathrm{H}}{\partial \underset{b}{\varphi}} - \frac{\partial}{\partial \mathrm{x}_k}\frac{\partial \mathrm{H}}{\partial \underset{b}{\varphi}_{,k}}\right]\frac{\partial \underset{b}{\varphi}}{\partial \mathrm{x}_i}\\
&+ \frac{\partial}{\partial \mathrm{x}_k}\left(\frac{\partial \mathrm{H}}{\partial \underset{a}{\mathrm{W}}_{j,k}}\frac{\partial \underset{a}{\mathrm{W}}_j}{\partial \mathrm{x}_i} + \frac{\partial \mathrm{H}}{\partial \underset{b}{\varphi}_{,k}}\frac{\partial \underset{b}{\varphi}}{\partial \mathrm{x}_i}\right) = \frac{\partial}{\partial \mathrm{x}_k}\left(\frac{\partial \mathrm{H}}{\partial \underset{a}{\mathrm{W}}_{j,k}}\frac{\partial \underset{a}{\mathrm{W}}_j}{\partial \mathrm{x}_i} + \frac{\partial \mathrm{H}}{\partial \underset{b}{\varphi}_{,k}}\frac{\partial \underset{b}{\varphi}}{\partial \mathrm{x}_i}\right)\\
&= \mathrm{div}_k\,(\nabla \underset{a}{\mathbf{W}}^T\cdot \underset{a}{\mathbf{S}} + \nabla \underset{b}{\varphi}\underset{b}{\boldsymbol{\phi}}),
\end{aligned}
$$

where we have used (3.3) and (3.5). The expression (5.3) follows directly.